3 Representing Relations

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero–one matrix. Suppose that R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$. (Here the elements of the sets A and B have been listed in a particular, but arbitrary, order. Furthermore, when $A = B$ we use the same ordering for A and B.) The relation R can be represented by the matrix $\mathbf{M}_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

In other words, the zero–one matrix representing R has a 1 as its (i, j) entry when a_i is related to b_j, and a 0 in this position if a_i is not related to b_j. (Such a representation depends on the orderings used for A and B.)

The use of matrices to represent relations is illustrated in Examples 1–6.

EXAMPLE 1 Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Solution: Because $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix for R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The 1s in $\mathbf{M}_R$ show that the pairs (2, 1), (3, 1), and (3, 2) belong to R. The 0s show that no other pairs belong to R. ◀

Proposition

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} \qquad \text{and} \qquad \mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}.$$

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S.$$

$$\mathbf{M}_{R^n} = \mathbf{M}_R^{[n]},$$

EXAMPLE 4 Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Solution: The matrices of these relations are

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

EXAMPLE 5 Find the matrix representing the relations $S \circ R$, where the matrices representing R and S are

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Solution: The matrix for $S \circ R$ is

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

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EXAMPLE 6 Find the matrix representing the relation R^2, where the matrix representing R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Solution: The matrix for R^2 is

$$\mathbf{M}_{R^2} = \mathbf{M}_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Equivalence Relations

DEFINITION 1 A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

DEFINITION 2 Two elements a and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

EXAMPLE 1 Let R be the relation on the set of integers such that aRb if and only if $a = b$ or $a = -b$. In Section 9.1 we showed that R is reflexive, symmetric, and transitive. It follows that R is an equivalence relation. ◀

EXAMPLE 2 Let R be the relation on the set of real numbers such that aRb if and only if $a - b$ is an integer. Is R an equivalence relation?

Extra Examples

Solution: Because $a - a = 0$ is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reflexive. Now suppose that aRb. Then $a - b$ is an integer, so $b - a$ is also an integer. Hence, bRa. It follows that R is symmetric. If aRb and bRc, then $a - b$ and $b - c$ are integers. Therefore, $a - c = (a - b) + (b - c)$ is also an integer. Hence, aRc. Thus, R is transitive. Consequently, R is an equivalence relation. ◀

EXAMPLE 3 **Congruence Modulo m** Let m be an integer with $m > 1$. Show that the relation

$$R = \{(a, b) \mid a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.

Solution: Recall from Section 4.1 that $a \equiv b \pmod{m}$ if and only if m divides $a - b$. Note that $a - a = 0$ is divisible by m, because $0 = 0 \cdot m$. Hence, $a \equiv a \pmod{m}$, so congruence modulo m is reflexive. Now suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, so $a - b = km$, where k is an integer. It follows that $b - a = (-k)m$, so $b \equiv a \pmod{m}$. Hence, congruence modulo m is symmetric. Next, suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Therefore, there are integers k and l with $a - b = km$ and $b - c = lm$. Adding these two equations shows that $a - c = (a - b) + (b - c) = km + lm = (k + l)m$. Thus, $a \equiv c \pmod{m}$. Therefore, congruence modulo m is transitive. It follows that congruence modulo m is an equivalence relation. ◀

EXAMPLE 7 Let R be the relation on the set of real numbers such that $x R y$ if and only if x and y are real numbers that differ by less than 1, that is $|x - y| < 1$. Show that R is not an equivalence relation.

Solution: R is reflexive because $|x - x| = 0 < 1$ whenever $x \in \mathbf{R}$. R is symmetric, for if $x R y$, where x and y are real numbers, then $|x - y| < 1$, which tells us that $|y - x| = |x - y| < 1$, so that $y R x$. However, R is not an equivalence relation because it is not transitive. Take $x = 2.8$, $y = 1.9$, and $z = 1.1$, so that $|x - y| = |2.8 - 1.9| = 0.9 < 1$, $|y - z| = |1.9 - 1.1| = 0.8 < 1$, but $|x - z| = |2.8 - 1.1| = 1.7 > 1$. That is, $2.8 \, R \, 1.9$, $1.9 \, R \, 1.1$, but $2.8 \not R \, 1.1$. ◀

Equivalence Classes

DEFINITION 3 Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is

$$[a]_R = \{s \mid (a, s) \in R\}.$$

If $b \in [a]_R$, then b is called a **representative** of this equivalence class. Any element of a class can be used as a representative of this class. That is, there is nothing special about the particular element chosen as the representative of the class.

EXAMPLE 8 What is the equivalence class of an integer for the equivalence relation of Example 1?

Solution: Because an integer is equivalent to itself and its negative in this equivalence relation, it follows that $[a] = \{-a, a\}$. This set contains two distinct integers unless $a = 0$. For instance, $[7] = \{-7,\ 7\}$, $[-5] = \{-5,\ 5\}$, and $[0] = \{0\}$. ◀

EXAMPLE 9 What are the equivalence classes of 0 and 1 for congruence modulo 4?

Solution: The equivalence class of 0 contains all integers a such that $a \equiv 0 \pmod{4}$. The intege in this class are those divisible by 4. Hence, the equivalence class of 0 for this relation is

$$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}.$$

The equivalence class of 1 contains all the integers a such that $a \equiv 1 \pmod{4}$. The integers in this class are those that have a remainder of 1 when divided by 4. Hence, the equivalence class of 1 for this relation is

$$[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}.$$

◀

In Example 9 the equivalence classes of 0 and 1 with respect to congruence modulo 4 were found. Example 9 can easily be generalized, replacing 4 with any positive integer m. The equivalence classes of the relation congruence modulo m are called the **congruence classes modulo** m. The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\ldots, a - 2m, a - m, a, a + m, a + 2m, \ldots\}$. For instance, from Example 9 it follows that $[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$ and $[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$.

Equivalence Classes and Partitions

Let R be a relation on the set A. Theorem 1 shows that the equivalence classes of two elements of A are either identical or disjoint.

THEOREM 1

Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(*i*) aRb (*ii*) $[a] = [b]$ (*iii*) $[a] \cap [b] \neq \emptyset$

Proof: We first show that (*i*) implies (*ii*). Assume that aRb. We will prove that $[a] = [b]$ by showing $[a] \subseteq [b]$ and $[b] \subseteq [a]$. Suppose $c \in [a]$. Then aRc. Because aRb and R is symmetric, we know that bRa. Furthermore, because R is transitive and bRa and aRc, it follows that bRc. Hence, $c \in [b]$. This shows that $[a] \subseteq [b]$. The proof that $[b] \subseteq [a]$ is similar; it is left as an exercise for the reader.

Second, we will show that (*ii*) implies (*iii*). Assume that $[a] = [b]$. It follows that $[a] \cap [b] \neq \emptyset$ because $[a]$ is nonempty (because $a \in [a]$ because R is reflexive).

Next, we will show that (*iii*) implies (*i*). Suppose that $[a] \cap [b] \neq \emptyset$. Then there is an element c with $c \in [a]$ and $c \in [b]$. In other words, aRc and bRc. By the symmetric property, cRb. Then by transitivity, because aRc and cRb, we have aRb.

Because (*i*) implies (*ii*), (*ii*) implies (*iii*), and (*iii*) implies (*i*), the three statements, (*i*), (*ii*), and (*iii*), are equivalent. ◁

THEOREM 2

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

EXAMPLE 14

What are the sets in the partition of the integers arising from congruence modulo 4?

Solution: There are four congruence classes, corresponding to $[0]_4$, $[1]_4$, $[2]_4$, and $[3]_4$. They are the sets

$$
\begin{aligned}
[0]_4 &= \{\ldots, -8, -4, 0, 4, 8, \ldots\},\\
[1]_4 &= \{\ldots, -7, -3, 1, 5, 9, \ldots\},\\
[2]_4 &= \{\ldots, -6, -2, 2, 6, 10, \ldots\},\\
[3]_4 &= \{\ldots, -5, -1, 3, 7, 11, \ldots\}.
\end{aligned}
$$

These congruence classes are disjoint, and every integer is in exactly one of them. In other words, as Theorem 2 says, these congruence classes form a partition. ◀

Partial Orderings

DEFINITION 1 A relation R on a set S is called a *partial ordering* or *partial order* if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

EXAMPLE 1 Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

Solution: Because $a \geq a$ for every integer a, $\geq$ is reflexive. If $a \geq b$ and $b \geq a$, then $a = b$. Hence, $\geq$ is antisymmetric. Finally, $\geq$ is transitive because $a \geq b$ and $b \geq c$ imply that $a \geq c$. It follows that $\geq$ is a partial ordering on the set of integers and $(\mathbf{Z}, \geq)$ is a poset. ◀

EXAMPLE 3 Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Solution: Because $A \subseteq A$ whenever A is a subset of S, $\subseteq$ is reflexive. It is antisymmetric because $A \subseteq B$ and $B \subseteq A$ imply that $A = B$. Finally, $\subseteq$ is transitive, because $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$. Hence, $\subseteq$ is a partial ordering on $P(S)$, and $(P(S), \subseteq)$ is a poset. ◀

DEFINITION 2 The elements a and b of a poset $(S, \preccurlyeq)$ are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called *incomparable*.

EXAMPLE 5 In the poset $(\mathbf{Z}^+, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Solution: The integers 3 and 9 are comparable, because $3 \mid 9$. The integers 5 and 7 are incomparable, because $5 \nmid 7$ and $7 \nmid 5$. ◀